

Principles of Communications

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Chapter 6: Signal Space Representation

Textbook: Chapter 8.1

Signal Space Concepts

- The key to analyzing and understanding the performance of digital transmission is the realization that
 - **signals used in communications can be expressed and visualized graphically**
- Thus, we need to understand **signal space concepts** as applied to digital communications

Traditional Bandpass Signal Representations

- **Baseband** signals are the message signal generated at the source
- **Passband signals** (also called **bandpass signals**) refer to the signals after modulating with a carrier. The bandwidth of these signals are usually **small** compared to the carrier frequency f_c
- Passband signals can be represented in three forms
 - ✓ Magnitude and Phase representation
 - ✓ Quadrature representation
 - Complex Envelop representation

Magnitude and Phase Representation

$$s(t) = a(t) \cos [2\pi f_c t + \theta(t)]$$

Where $a(t)$ is the magnitude of $s(t)$
and $\theta(t)$ is the phase of $s(t)$

Quadrature or I/Q Representation

$$s(t) = x(t) \cos(2\pi f_c t) - y(t) \sin(2\pi f_c t)$$

where $x(t)$ and $y(t)$ are real-valued **baseband** signals called the **in-phase** and **quadrature components** of $s(t)$.

Signal space is a more convenient way than I/Q representation to study modulation scheme

Vectors and Space

- Consider an **n-dimensional space** with unity basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ (think of the x-y-z axis in a coordinate system)
- Any **vector** \mathbf{a} in the space can be written as

$$\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}_i \quad \Rightarrow \quad \mathbf{a} = (a_1, a_2, \dots, a_n)$$

$n \triangleq$ **Dimension** = **Minimum number of vectors** that is necessary and sufficient for representation of any vector in space

- Definitions:

- **Inner Product** $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$

- **a** and **b** are **Orthogonal** if $\mathbf{a} \cdot \mathbf{b} = 0$

- $\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^n a_i^2} = \text{Norm of } \mathbf{a}$

- A set of vectors are **orthonormal** if they are mutually orthogonal and all have unity norm

Basis Vectors

- The set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of a space are chosen such that:
 - Should be **complete** or **span** the vector space: **any vector a can be expressed as a linear combination of these vectors.**
 - Each basis vector should be **orthogonal** to all others

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, \quad \forall i \neq j$$

- Each basis vector should be **normalized**: $\|\mathbf{e}_i\| = 1, \quad \forall i$
- A set of basis vectors satisfying these properties is also said to be a **complete orthonormal basis**
- In an **n-dim** space, we can have **at most n basis vectors**

Signal Space

- Basic Idea: If a signal can be represented by **n-tuple**, then it can be treated in much the same way as a **n-dim vector**.
- Let $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ be n signals
- Consider a signal $x(t)$ and suppose that

$$x(t) = \sum_{i=1}^n \phi_i(t)$$

- If every signal can be written as above \Rightarrow $\{\phi_1(t), \dots, \phi_n(t)\} \sim$ **basis functions** and we have a **n-dim signal space**

Orthonormal Basis

- Signal set $\{\phi_k(t)\}^n$ is an **orthogonal** set if

$$\int_{-\infty}^{\infty} \phi_j(t) \phi_k(t) dt = \begin{cases} 0 & j \neq k \\ c_j & j = k \end{cases}$$

- If $c_j = 1 \quad \forall_j \Rightarrow \{\phi_k(t)\}$ is an **orthonormal** set.
- In this case,

$$x_k = \int_{-\infty}^{\infty} x(t) \phi_k(t) dt$$

$$x(t) = \sum_{i=1}^n x_i \phi_i(t)$$

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

Key Property

- Given the set of the orthonormal basis

$$\{\phi_1(t), \dots, \phi_n(t)\}$$

- Let $x(t)$ and $y(t)$ be represented as

$$x(t) = \sum_{i=1}^n x_i \phi_i(t) , \quad y(t) = \sum_{i=1}^n y_i \phi_i(t)$$

$$\text{with } \mathbf{x} = (x_1, x_2, \dots, x_n), \quad \mathbf{y} = (y_1, y_2, \dots, y_n)$$

- Then the **inner product** of \mathbf{x} and \mathbf{y} is

$$\mathbf{x} \cdot \mathbf{y} = \int_{-\infty}^{\infty} x(t)y(t)dt$$

- Proof

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)y(t)dt &= \int_{-\infty}^{\infty} \left[\sum_{i=1}^n x_i \phi_i(t) \right] \left[\sum_{j=1}^n y_j \phi_j(t) \right] dt \\ &= \sum_{k=1}^n x_k y_k \triangleq \mathbf{x} \cdot \mathbf{y}\end{aligned}$$

Since $\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

➤ $E_x = \text{Energy of } \mathbf{x}(t) = \int_{-\infty}^{\infty} x^2(t)dt$



$$E_x = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$$

Basis Functions for a Signal Set

- Consider a set of M signals (M-ary symbol) $\{s_i(t), i=1,2,\dots,M\}$ with finite energy. That is

$$\int_{-\infty}^{\infty} s_i^2(t) dt < \infty$$

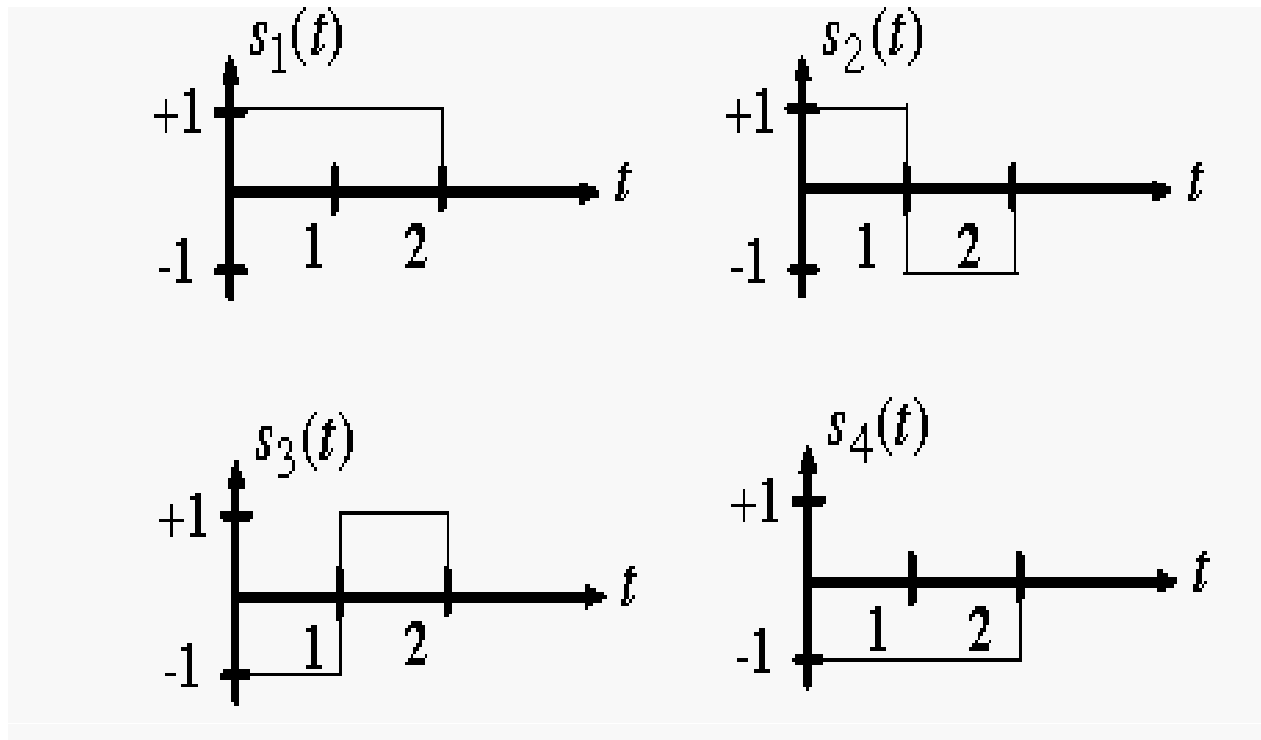
- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t) \quad \text{for } i = 1, \dots, M$$

where $N \leq M$ is the **dimension** of the signal space and $\{\phi_j(t)\}_1^N$ are called the **orthonormal basis functions**

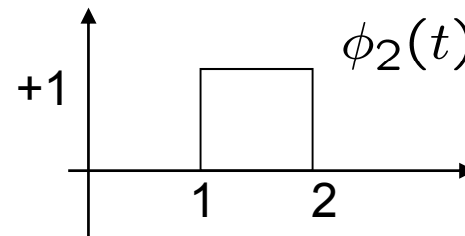
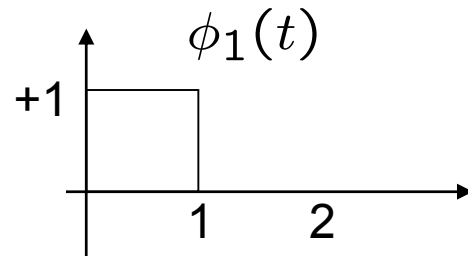
Example 1

Consider the following signal set:



Example 1 (Cont'd)

- By inspection, the signals can be expressed in terms of the following two basis functions:



$$s_1(t) = 1 \cdot \phi_1(t) + 1 \cdot \phi_2(t)$$

$$s_3(t) = -1 \cdot \phi_1(t) + 1 \cdot \phi_2(t)$$

$$s_2(t) = 1 \cdot \phi_1(t) - 1 \cdot \phi_2(t)$$

$$s_4(t) = -1 \cdot \phi_1(t) - 1 \cdot \phi_2(t)$$

- Not that the basis is orthogonal $\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = 0$
- Also note that each these functions have unit energy

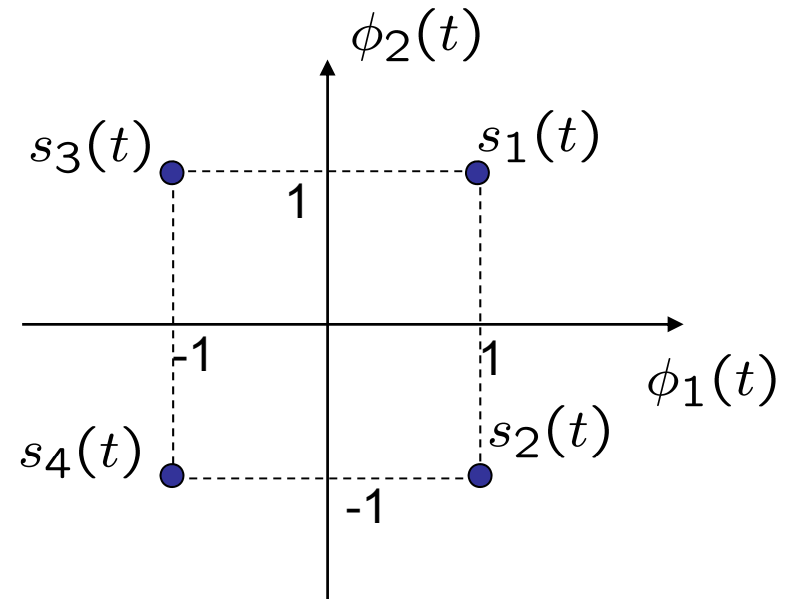
$$\int_{-\infty}^{\infty} |\phi_1(t)|^2 dt = \int_{-\infty}^{\infty} |\phi_2(t)|^2 dt = 1$$

- We say that they form an orthonormal basis

Example 1 (Cont'd)

Constellation diagram:

- A representation of a **digital modulation** scheme in the **signal space**
- Axes are labeled with $\phi_1(t)$ and $\phi_2(t)$
- Possible signals are plotted as points, called **constellation points**



Example 2

- Suppose our signal set can be represented in the following form

$$s(t) = \pm \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \pm \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

with $t \in [0, T)$ and $f_c T \gg 1$

- We can choose the basis functions as follows

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

$$t \in [0, T)$$

Example 2 (Cont'd)

- Since

$$\begin{aligned}\int_0^T \phi_1(t)\phi_2(t)dt &= \int_0^T \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \cdot \sqrt{\frac{2}{T}} \sin(2\pi f_c t) dt \\ &= \frac{2}{T} \int_0^T \frac{1}{2} [\sin(0) + \sin(4\pi f_c t)] dt \\ &= \frac{-1}{4\pi f_c T} [\cos(4\pi f_c t)]_0^T \approx 0, \text{ for } f_c T \gg 1\end{aligned}$$

and

$$\int_0^T |\phi_1(t)|^2 dt = \int_0^T |\phi_2(t)|^2 dt = \frac{2}{T} \int_0^T \frac{1}{2} [1 + \cos(4\pi f_c t)] dt \approx 1$$

- The basis functions are thus orthogonal and they are also normalized

Example 2 (Cont'd)

- These basis functions are quite common and can describe various modulation schemes
- Example 2 is **QPSK modulation**. Its constellation diagram is identical to Example 1

Notes on Signal Space

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set
- In previous examples, we were able to guess the correct basic functions
- In general, is there any method which allows us to find a complete orthonormal basis for an arbitrary signal set?
 - Gram-Schmidt Orthogonalization (GSO) Procedure

Gram Schmidt Orthogonalization (GSO) Procedure

- Suppose we are given a signal set

$$\{s_1(t), \dots, s_M(t)\}$$

- Find the orthogonal basis functions for this signal set

$$\{\phi_1(t), \dots, \phi_K(t)\}$$

where $K \leq M$

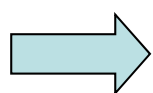
Step 1: Construct the First Basis Function

- Compute the energy in signal 1:

$$E_1 = \int_{-\infty}^{\infty} s_1^2(t) dt$$

- The first basis function is just a normalized version of $s_1(t)$

$$\phi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$



$$s_1(t) = s_{11} \phi_1(t) = \sqrt{E_1} \phi_1(t)$$

$$s_{11} = \int_{-\infty}^{\infty} s_1(t) \phi_1(t) dt = \sqrt{E_1}$$

Step 2: Construct the Second Basis Function

- Compute correlation between signal 2 and basic function 1

$$s_{21} = \int_{-\infty}^{\infty} s_2(t)\phi_1(t)dt$$

- Subtract off the correlation portion

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \quad \longrightarrow \quad g_2(t) \text{ is orthogonal to } \phi_1(t)$$

- Compute the energy in the remaining portion

$$E_{g_2} = \int_{-\infty}^{\infty} [g_2(t)]^2 dt$$

- Normalize the remaining portion

$$\phi_2(t) = \frac{1}{\sqrt{E_{g_2}}}g_2(t)$$



$$\longrightarrow s_{22} = \int_{-\infty}^{\infty} s_2(t)\phi_2(t)dt = \sqrt{E_{g_2}}$$

Step 3: Construct Successive Basis Functions

- For signal $s_k(t)$, compute $s_{ki} = \int_{-\infty}^{\infty} s_k(t)\phi_i(t)dt$

- Define $g_k(t) = s_k(t) - \sum_{i=1}^{k-1} s_{ki}\phi_i(t)$

- Energy of $g_k(t)$: $E_{g_k} = \int_{-\infty}^{\infty} [g_k(t)]^2 dt$

- k -th basis function: $\phi_k(t) = \frac{1}{\sqrt{E_{g_k}}}g_k(t)$

- In general

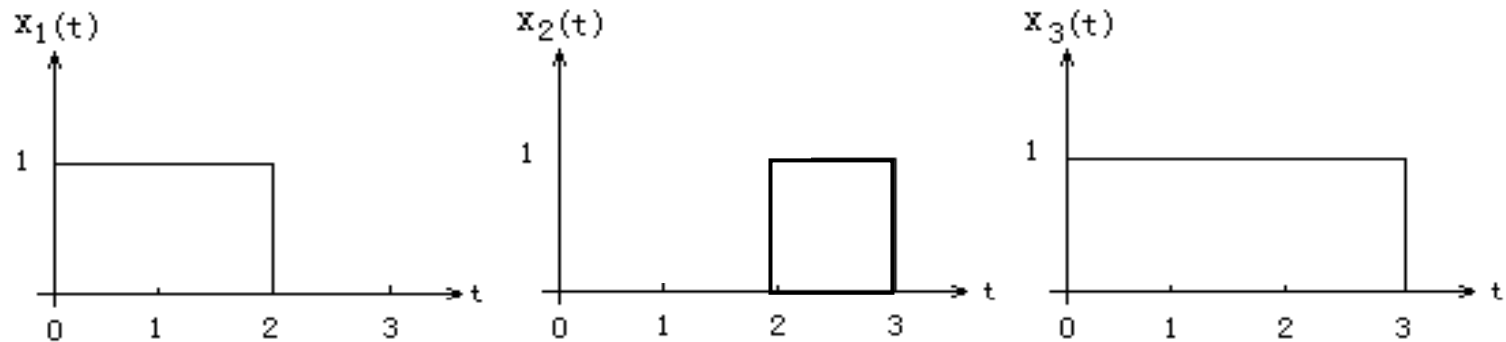
$$s_{kk} = \int_{-\infty}^{\infty} s_k(t)\phi_k(t)dt = \sqrt{E_{g_k}}$$

Summary of GSO Procedure

- 1st basis function is normalized version of the first signal
- Successive basis functions are found by removing portions of signals that are correlated to previous basis functions and normalizing the result
- This procedure is repeated until all basis functions are found
- If $g_k(t) = 0$, no new basis functions is added
- The order in which signals are considered is arbitrary

Example: GSO

- 1) Use the Gram-Schmidt procedure to find a set orthonormal basis functions corresponding to the signals show below



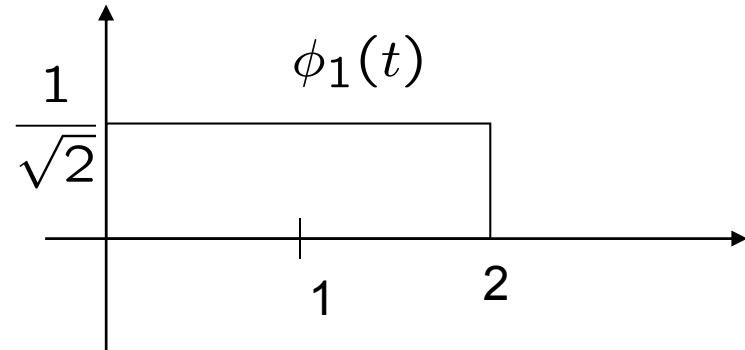
- 2) Express x_1 , x_2 , and x_3 in terms of the orthonormal basis functions found in part 1)
- 3) Draw the constellation diagram for this signal set

Solution: 1)

Step 1: $E_1 = \int_{-\infty}^{\infty} x_1^2(t) dt = 2$

$$\phi_1(t) = \frac{1}{\sqrt{2}} x_1(t)$$

$$x_{11} = \sqrt{2}$$

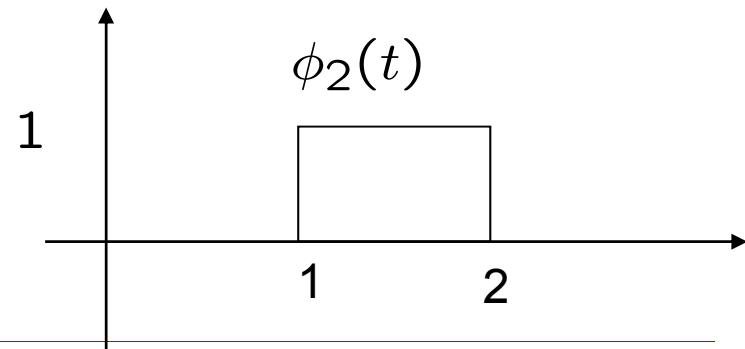


Step 2: $x_{21} = \int_{-\infty}^{\infty} x_2(t) \phi_1(t) dt = 0$

$$g_2(t) = x_2(t) \text{ and } E_{g_2} = 1$$

$$\phi_2(t) = x_2(t)$$

$$x_{22} = 1$$



Solution: 1) (Cont'd)

- Step 3: $x_{31} = \int_{-\infty}^{\infty} x_3(t)\phi_1(t)dt = \sqrt{2}$

$$x_{32} = \int_{-\infty}^{\infty} x_3(t)\phi_2(t)dt = 1$$

$$g_3(t) = x_3(t) - x_{31}f_1(t) - x_{32}f_2(t) = 0$$

=> No more new basis functions

Procedure completes

$$\begin{cases} \phi_1(t) = \frac{1}{\sqrt{2}}x_1(t) \\ \phi_2(t) = x_2(t) \end{cases}$$

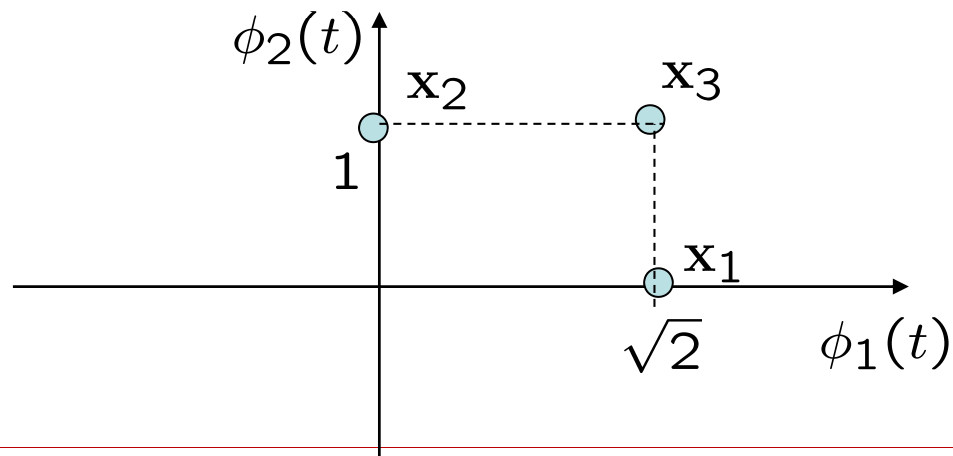
Solution: 2) and 3)

- Express x_1 , x_2 , x_3 in basis functions

$$x_1(t) = \sqrt{2}\phi_1(t), \quad x_2(t) = \phi_2(t)$$

$$x_3(t) = \sqrt{2}\phi_1(t) + \phi_2(t)$$

- Constellation diagram



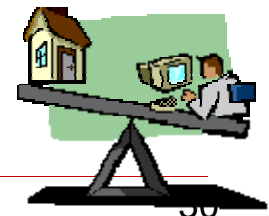
Exercise

Given a set of signals (8PSK modulation)

$$s_i(t) = A \cos \left(2\pi f_c t + \frac{\pi}{4} i \right)$$

$$i = 0, 1, \dots, 7 \quad \text{and} \quad 0 \leq t < T$$

- Find the orthonormal basis functions using Gram Schmidt procedure
- What is the dimension of the resulting signal space ?
- Draw the constellation diagram of this signal set



Notes on GSO Procedure

- A signal set may have many different sets of basis functions
- A change of basis functions is essentially a rotation of the signal points around the origin.
- The order in which signal are used in the GSO procedure affect the resulting basis functions
- The choice of basis functions does not affect the performance of the modulation scheme

